Problem 16.16

Let $f(\mathbf{r})$ be any spherically symmetric function; that is, when expressed in spherical polar coordinates, (r, θ, ϕ), it has the form $f(\mathbf{r}) = f(r)$, independent of θ and ϕ. **(a)** Starting from the definition (16.38) of ∇^2, prove that

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf).$$

(b) Prove the same result using the formula inside the back cover for ∇^2 in spherical polar coordinates. (Obviously, this second proof is much simpler, but the hard work is hidden in the derivation of the formula for ∇^2.)

Solution

Part (a)

Start with definition (16.38).

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{16.38}$$

As a result,

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right).\end{aligned}$$

At the moment, f is in Cartesian coordinates: $f = f(x, y, z)$. Now switch to spherical coordinates (r, θ, ϕ), where θ is the angle from the polar axis.

$$\begin{cases} x = r\cos\phi\sin\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\theta \end{cases} \quad \Longleftrightarrow \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

f is now in spherical coordinates: $f = f(r, \theta, \phi)$. Use the chain rule to write the derivatives in terms of these new variables.

$$\begin{aligned}\nabla^2 f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \underbrace{\frac{\partial f}{\partial \theta}}_{=0}\frac{\partial \theta}{\partial x} + \underbrace{\frac{\partial f}{\partial \phi}}_{=0}\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \underbrace{\frac{\partial f}{\partial \theta}}_{=0}\frac{\partial \theta}{\partial y} + \underbrace{\frac{\partial f}{\partial \phi}}_{=0}\frac{\partial \phi}{\partial y}\right) \\ + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \underbrace{\frac{\partial f}{\partial \theta}}_{=0}\frac{\partial \theta}{\partial z} + \underbrace{\frac{\partial f}{\partial \phi}}_{=0}\frac{\partial \phi}{\partial z}\right)\end{aligned}$$

Because of radial symmetry, $f = f(r)$, so most of the terms vanish.

$$\nabla^2 f = \frac{\partial}{\partial x}\left(\frac{df}{dr}\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dr}\frac{\partial r}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{df}{dr}\frac{\partial r}{\partial z}\right)$$

In spherical coordinates $r = \sqrt{x^2 + y^2 + z^2}$, so

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot \frac{\partial}{\partial x}(x^2 + y^2 + z^2) = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot (2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot \frac{\partial}{\partial y}(x^2 + y^2 + z^2) = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot (2y) = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot \frac{\partial}{\partial z}(x^2 + y^2 + z^2) = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot (2z) = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{r}.$$

Consequently,

$$\begin{aligned}
\nabla^2 f &= \frac{\partial}{\partial x}\left(\frac{df}{dr}\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dr}\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{df}{dr}\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dr}\right)\frac{x}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial y}\left(\frac{df}{dr}\right)\frac{y}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial z}\left(\frac{df}{dr}\right)\frac{z}{\sqrt{x^2+y^2+z^2}} \\
&\qquad + \frac{df}{dr}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{df}{dr}\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{df}{dr}\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dr}\right)\frac{x}{r} + \frac{\partial}{\partial y}\left(\frac{df}{dr}\right)\frac{y}{r} + \frac{\partial}{\partial z}\left(\frac{df}{dr}\right)\frac{z}{r} \\
&\qquad + \frac{df}{dr}\left[\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{df}{dr}\left[\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{df}{dr}\left[\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}\right] \\
&= \frac{x}{r}\frac{\partial}{\partial x}\left(\frac{df}{dr}\right) + \frac{y}{r}\frac{\partial}{\partial y}\left(\frac{df}{dr}\right) + \frac{z}{r}\frac{\partial}{\partial z}\left(\frac{df}{dr}\right) + \frac{df}{dr}\left[\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}\right] \\
&= \frac{x}{r}\frac{\partial}{\partial x}\left(\frac{df}{dr}\right) + \frac{y}{r}\frac{\partial}{\partial y}\left(\frac{df}{dr}\right) + \frac{z}{r}\frac{\partial}{\partial z}\left(\frac{df}{dr}\right) + \frac{df}{dr}\left(\frac{2}{r}\right) \\
&= \frac{x}{r}\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}\right)\left(\frac{df}{dr}\right) + \frac{y}{r}\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}\right)\left(\frac{df}{dr}\right) \\
&\qquad + \frac{z}{r}\left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}\right)\left(\frac{df}{dr}\right) + \frac{2}{r}\frac{df}{dr} \\
&= \frac{x}{r}\left[\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{df}{dr}\right) + \frac{\partial \theta}{\partial x}\underbrace{\frac{\partial}{\partial \theta}\left(\frac{df}{dr}\right)}_{=0} + \frac{\partial \phi}{\partial x}\underbrace{\frac{\partial}{\partial \phi}\left(\frac{df}{dr}\right)}_{=0}\right] + \frac{y}{r}\left[\frac{\partial r}{\partial y}\frac{\partial}{\partial r}\left(\frac{df}{dr}\right) + \frac{\partial \theta}{\partial y}\underbrace{\frac{\partial}{\partial \theta}\left(\frac{df}{dr}\right)}_{=0} + \frac{\partial \phi}{\partial y}\underbrace{\frac{\partial}{\partial \phi}\left(\frac{df}{dr}\right)}_{=0}\right] \\
&\qquad + \frac{z}{r}\left[\frac{\partial r}{\partial z}\frac{\partial}{\partial r}\left(\frac{df}{dr}\right) + \frac{\partial \theta}{\partial z}\underbrace{\frac{\partial}{\partial \theta}\left(\frac{df}{dr}\right)}_{=0} + \frac{\partial \phi}{\partial z}\underbrace{\frac{\partial}{\partial \phi}\left(\frac{df}{dr}\right)}_{=0}\right] + \frac{2}{r}\frac{df}{dr} \\
&= \frac{x}{r}\left(\frac{x}{r}\frac{d^2 f}{dr^2}\right) + \frac{y}{r}\left(\frac{y}{r}\frac{d^2 f}{dr^2}\right) + \frac{z}{r}\left(\frac{z}{r}\frac{d^2 f}{dr^2}\right) + \frac{2}{r}\frac{df}{dr}.
\end{aligned}$$

Simplify the result.

$$\begin{aligned}
\nabla^2 f &= \left(\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right)\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} \\
&= \left(\frac{x^2 + y^2 + z^2}{r^2}\right)\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} \\
&= \left(\frac{r^2}{r^2}\right)\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} \\
&= \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} \\
&= \frac{1}{r}\left(r\frac{d^2 f}{dr^2} + 2\frac{df}{dr}\right) \\
&= \frac{1}{r}\left(r\frac{d^2 f}{dr^2} + \frac{df}{dr} + \frac{df}{dr}\right) \\
&= \frac{1}{r}\left[\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{df}{dr}\right] \\
&= \frac{1}{r}\frac{d}{dr}\left[r\frac{df}{dr} + f(r)\right] \\
&= \frac{1}{r}\frac{d}{dr}\left[\frac{d}{dr}[rf(r)]\right]
\end{aligned}$$

Therefore, for a radially symmetric function $f = f(r)$,

$$\nabla^2 f = \frac{1}{r}\frac{d^2}{dr^2}(rf).$$

Part (b)

Start with the formula for the Laplacian operator in the back of the book and use the fact that f is only a function of r.

$$\begin{aligned}
\nabla^2 f &= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \underbrace{\frac{\partial f}{\partial\theta}}_{=0}\right) + \frac{1}{r^2\sin^2\theta}\underbrace{\frac{\partial^2 f}{\partial\phi^2}}_{=0} \\
&= \frac{1}{r}\frac{d^2}{dr^2}(rf)
\end{aligned}$$